KERALA SCHOOL OF MATHEMATICS

MID-SEMESTER EXAMINATION - IV

ANALYSIS ON MANIFOLDS

November 16, 2025

Duration: 3 hours | Maximum Marks: 30

Name:

Problem	1	2	3	4	Total
Marks Obtained					

Problem 1. Let M be a smooth n–manifold and let $A \subset M$ be a closed subset with empty interior. Construct two smooth atlas structures $\mathcal{A}$ and $\mathcal{B}$ on M satisfying:

(a) $\mathcal{A}$ and $\mathcal{B}$ induce the same smooth structure on $M \setminus A$, but differ on every neighborhood of each point of A; [2]

(b) the identity map $\mathrm{id}_M : (M, \mathcal{A}) \to (M, \mathcal{B})$ is a homeomorphism but not a diffeomorphism; [2]

(c) there exists a diffeomorphism $F : (M, \mathcal{A}) \to (M, \mathcal{B})$. [2]

Give an explicit construction for the case $M = \mathbb{R}^n$ and $A = \{0\}$.

Problem 2. Let M and N be smooth manifolds. A map $F : M \to N$ is called *curve-smooth* if for every smooth curve $\gamma : \mathbb{R} \to M$, the composition $F \circ \gamma$ is a smooth curve in N.

(a) Show that every smooth map $F : M \to N$ is curve-smooth. [2]

(b) Prove that if $\dim M = 1$, then every curve-smooth map $F : M \to N$ is smooth. [2]

(c) For every $n \geq 2$, construct an explicit example of a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \circ \gamma$ is smooth for every smooth analytic curve $\gamma : \mathbb{R} \to \mathbb{R}^n$, but f itself is not smooth. ~~[2]~~
Boman's Theorem : A function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth if and only if $f \circ \gamma$ is smooth for every smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^n$.

Problem 3. Let M be a smooth manifold with or without boundary and p be a point of M. This problem asks you to prove the equivalence of the three common definitions of the tangent space T_pM.

(a) Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\phi : \mathcal{D}_pM \to T_pM$ by $(\phi v)f = v([f]_p)$. Show that ϕ is an isomorphism. [3]

(b) Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ starting at p (i.e., $\gamma(0) = p$) under the relation $\sim$ defined by:

$$\gamma_1 \sim \gamma_2 \quad \text{if} \quad (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. [3]

Problem 4. We recall the **Inverse Function Theorem for Manifolds**: *If $F : M \to N$ is a C^∞ map, $\dim M = \dim N$, and at a point $p \in M$, the differential $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, then F is a local diffeomorphism at p.*

A set $\{x_1, \dots, x_l\}$ of C^∞ functions defined on some neighborhood of p in M is called an **independent set at** p if the differentials $\{dx_1|_p, \dots, dx_l|_p\}$ form a linearly independent set in T_p^*M. You will now use this to prove foundational results about local coordinates.

(a) Let M be an n-dimensional manifold and $p \in M$. Let $\{x_1, \dots, x_k\}$ be an independent set of C^∞ functions at p, with $k \leq n$. Prove that this set can be extended to a local coordinate system $\{x_1, \dots, x_n\}$ on a neighborhood of p. Conclude that if $k = n$, the original set $\{x_1, \dots, x_n\}$ already forms a local coordinate system at p. [3]

(b) Let $F : M \to N$ be a C^∞ map, where $\dim M = n$ and $\dim N = m$. Assume that at $p \in M$, the differential $dF_p : T_pM \to T_{F(p)}N$ is **surjective**. Let $\{y_1, \dots, y_m\}$ form a coordinate system on some neighborhood of $F(p)$. Prove that the pullback functions $y_1 \circ F, \dots, y_m \circ F$ form part of a coordinate system on some neighborhood of p. [3]

(c) Let M be an n-dimensional manifold. Suppose that $\{x_1, \dots, x_k\}$ is a set of C^∞ functions on a neighborhood of $p \in M$ such that their differentials $\{dx_1|_p, \dots, dx_k|_p\}$ span the cotangent space T_p^*M. Prove that a subset of these functions forms a coordinate system on a neighborhood of p. [3]

(d) Let $F : M \to N$ be a C^∞ map, where $\dim M = n$ and $\dim N = m$. Assume that at $p \in M$, the differential $dF_p : T_pM \to T_{F(p)}N$ is **injective**. Let $\{y_1, \dots, y_m\}$ form a coordinate system on a neighborhood of $F(p)$. Prove that a subset of the functions $\{y_j \circ F\}$ forms a coordinate system on a neighborhood of p. Conclude that F is one-to-one on a (possibly smaller) neighborhood of p. [3]