KERALA SCHOOL OF MATHEMATICS

MID-SEMESTER EXAMINATION - III

ANALYSIS ON MANIFOLDS

October 10, 2025

Duration: 3 hours | Maximum Marks: 30

Name:

Problem	1	2	3	4	Total
Marks Obtained					

Problem 1. Let

$$\widetilde{\Sigma} = \{\, \mathbf{f}(u,v) = (e^u \cos v,\ e^u \sin v,\ v) \ :\ (u,v) \in \mathbf{R}^2 \,\} \subset \mathbf{R}^3$$

and let

$$\Sigma = \{\, (x,y,0) \in \mathbf{R}^3 : (x,y) \neq (0,0) \,\}$$

be the punctured xy-plane. Define $p : \widetilde{\Sigma} \to \Sigma$ by the vertical projection

$$p(x,y,z) = (x,y,0),$$

so that $p(\mathbf{f}(u,v)) = (e^u \cos v, e^u \sin v, 0)$.

(a) Show that both $\widetilde{\Sigma}$ and Σ are smooth 2-manifolds in $\mathbf{R}^3$. [3]

(b) Prove that p is smooth and surjective, and that for every $q \in \widetilde{\Sigma}$ there exists an open neighborhood $U \subset \widetilde{\Sigma}$ such that

$$p : U \to p(U)$$

is a diffeomorphism onto an open subset of Σ. [3]

(c) Explain briefly why p is not a global diffeomorphism. [2]

Problem 2. Let

$$F(x,y,z) = x^2 + y^2 + e^z - 4, \qquad F : \mathbf{R}^3 \to \mathbf{R},$$

and define

$$M = \{(x,y,z) \in \mathbf{R}^3 : F(x,y,z) \leq 0\}.$$

(a) Show that F is a smooth defining function of the domain interior(M) in $\mathbf{R}^3$. [4]

(b) Prove that M is a 3-dimensional manifold with boundary and describe ∂M explicitly. Describe coordinate charts near a boundary point of M. [3]

Problem 3. This problem asks you to first prove a generalization of the change of variables formula for maps that are not one-to-one, and then apply it to compute a specific integral.

(a) Suppose $\phi : \Omega \to \Omega'$ is a smooth, **proper map** between open sets in $\mathbb{R}^n$. A map is proper if the preimage of every compact set is compact. Assume that for almost every $y \in \Omega'$, the preimage set $\phi^{-1}(y)$ consists of exactly N distinct points. Prove that for any integrable function $f : \Omega' \to \mathbb{R}$, the correct change of variables formula is:

$$\int_{\Omega'} f(y)\, dy = \frac{1}{N} \int_{\Omega} f(\phi(x)) |\det(D\phi(x))|\, dx$$

[4]

(b) Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\phi(x, y) = (x^2 - y^2,\, 2xy)$$

Let the domain $\Omega \subset \mathbb{R}^2$ be the annulus defined by:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$$

Compute the value of the integral:

$$\iint_{\Omega'} \frac{1}{\sqrt{u^2 + v^2}}\, du\, dv.$$

Here $\Omega' = \phi(\Omega)$. [3]

Problem 4. Let $M = g^{-1}(c)$ be a smooth $(n-1)$-manifold, where $g : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function and c is a regular value. Assume that near a point $p \in M$ where $\frac{\partial g}{\partial x_n} \neq 0$, the manifold is the graph of a function $x_n = h(x_1, \ldots, x_{n-1})$.

(a) By considering the projection of the surface onto the $x_1, \ldots, x_{n-1}$ hyperplane, prove that the surface measure dV_M is given by:

$$dV_M = \frac{|\nabla g|}{|\frac{\partial g}{\partial x_n}|}\, dx_1 \ldots dx_{n-1}$$

[4]

(b) Let M_4 be the portion of the surface $x^2 + y^2 - z = 0$ for which $z \leq 4$. Using the formula from part (a), compute the integral:

$$\int_{M_4} z\, dV_M$$

[3]

(c) Let M be the graph of a C^1 function $h : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n-1}$ is open. This surface can be described parametrically by $\alpha(u) = (u, h(u))$ and implicitly by the level set of $g(x) = h(x_1, \ldots, x_{n-1}) - x_n = 0$. Prove that the volume measures derived from these two descriptions are identical by showing that for all $u \in U$:

$$\sqrt{\det(D\alpha(u)^T D\alpha(u))} = \frac{|\nabla g(\alpha(u))|}{|\frac{\partial g}{\partial x_n}(\alpha(u))|}$$

[4]