Kerala School of Mathematics

Mid-semester Examination - II

Analysis on Manifolds

September 29, 2025

Duration: 2 hours — Maximum marks: 30

Problem 1. Let $A \subset \mathbb{R}^n$ be a closed set, and set $U = \mathbb{R}^n \setminus A$.

(a) Show that there exists an exhaustion of U by relatively compact open sets, i.e.,

$$U_1 \subset U_2 \subset U_3 \subset \cdots \subset U, \qquad \bigcup_{k=1}^{\infty} U_k = U,$$

such that $\overline{U_k}$ is compact and $\overline{U_k} \subset U_{k+1}$ for all k. [2]

(b) Let $\{\varphi_k\}$ be a smooth partition of unity subordinate to the cover $\{U_{k+1} \setminus \overline{U_{k-1}}\}_{k\geq 1}$ (with $U_0 = \emptyset$). Define smooth functions $\psi_k = 2^{-k}\varphi_k$ and set

$$F(x) = \sum_{k=1}^{\infty} \psi_k(x), \qquad x \in U.$$

Show that this sum is well-defined and smooth on U. [2]

(c) Extend F to all of $\mathbb{R}^n$ by setting $F(x) = 0$ for $x \in A$. Prove that F is smooth at every point $p \in A$, i.e., for every multi-index α,

$$\lim_{x\to p} D^\alpha F(x) = 0.$$

[3]

Problem 2. Let A be an invertible $n \times n$ real matrix, and consider the set

$$E = \{x \in \mathbb{R}^n : x = Ay,\, y \in B_1(0)\},$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^n$. The volume of the unit ball is

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)},$$

where the Gamma function Γ is defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

(a) Using the Change of Variables formula, prove that

$$m(E) = |\det A|\, V_n.$$

[3]

(b) Deduce the volume of the ellipsoid

$$E' = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1 \right\}$$

by choosing an appropriate diagonal matrix A. [3]

Problem 3. Let $A = [0,1]^2 \subset \mathbf{R}^2$ and define a C^1 map $\alpha : A \to \mathbf{R}^4$ by

$$\alpha(u, v) = (u, v, u^2 v, \sin(\pi u v)).$$

(a) Compute the derivative matrix $D\alpha(u, v)$ and write the 2-volume element

$$|V(D\alpha(u, v))| = \sqrt{\det\left((D\alpha)^T D\alpha\right)}$$

explicitly as a function of u and v. [2]

(b) Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be the linear map

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Show that for each $(u, v) \in A$, the 2-volume element of the transformed surface satisfies

$$|V(D(T \circ \alpha)(u, v))| = \sqrt{\det\left((TD\alpha(u, v))^T (TD\alpha(u, v))\right)}.$$

[3]

(c) Let $Z = T(Y_\alpha)$. Show that the total 2-volume of Z can be expressed as

$$v(Z) = \int_A \sqrt{\det\left((D(T \circ \alpha)(u, v))^T \, D(T \circ \alpha)(u, v)\right)} \, du \, dv.$$

Further, prove that this integral can be factored as

$$v(Z) = \int_A \sqrt{\det\left((D\alpha(u, v))^T D\alpha(u, v)\right)} \, \sqrt{\det\left((DT|_{\mathrm{Im}\ D\alpha(u,v)})^T DT|_{\mathrm{Im}\ D\alpha(u,v)}\right)} \, du \, dv,$$

and conclude that $v(Z)$ is independent of the choice of parametrization α of Y_α. [3]

Problem 4. Consider the surface

$$M = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1, \ x^2 + y^2 = z\}.$$

(a) Show that M is a 1-dimensional manifold (a curve) in $\mathbf{R}^3$ by checking that the gradients of the defining functions are linearly independent at every point of M. [3]

(b) Use the Implicit Function Theorem to express y and z locally as smooth functions of x (or any other suitable choice of parameter). [3]

(c) Let $f(x, y, z) = z^2$. Set up the integral

$$\int_M f \, dV$$

as an integral over your chosen parameter. Express the integrand in terms of this parameter using the chain rule. You do not need to compute the integral explicitly. [3]